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On the Poincaré Problem

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We derive some restrictions on the possible degrees of algebraic invariant curves and on the possible form of algebraic integrating factors, for plane polynomial vector fields whose stationary points at infinity satisfy a certain genericity condition. The method is elementary, and we show by example that it also yields strong

1. INTRODUCTION AND PRELIMINARIES

The question of the existence of algebraic invariant curves and algebraic integrating factors of plane polynomial vector fields goes back to Darboux and Poincaré; see Schlomiuk [25, 26] for remarks on the history of the problem. They noted that, given a polynomial vector field f on $\mathbf{C}^2$, the question can be answered by elementary means as soon as an upper bound for the possible degrees of irreducible invariant algebraic curves is established. The Poincaré problem to be discussed in this article consists of finding such bounds, and possibly invariant curves and integrating factors. (There are other versions of this problem; many authors discuss it in the context of holomorphic foliations of the projective plane.)

It should be emphasized that this problem is interesting for several reasons. Prelle and Singer [21] showed that the existence of an algebraic integrating factor is a necessary condition for the existence of an elementary first integral. Moreover, they showed that deciding about the existence of an algebraic integrating factor ist the only obstacle to deciding about the existence of an elementary first integral. In his classical work [13], Lie established a correspondence between nontrivial infinitesimal symmetries and integrating factors of two-dimensional vector fields. Therefore, to determine an integrating factor also means to solve the nontrivial (and not algorithmically accessible) problem of finding a nontrivial local one-parameter symmetry group. Finally, the existence of an algebraic integrating factor has a strong influence on the qualitative behavior of a (real)

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differential equation; see Schlomiuk [24, 25, 26] and Christopher [7] for applications to the center problem.

While it is known from a theorem of Darboux that for every polynomial vector field there exists an upper bound for the possible degrees of irreducible invariant algebraic curves, it is a hard problem to determine such an upper bound explicitly for any given vector field. There has been substantial progress in recent years, and bounds have been determined under additional conditions on the invariant curves or on the stationary points of the vector fields; cf. Cerveau and Lins Neto [6], and Carnicer [5]. In particular, Carnicer found degree bounds whenever the vector field has no dicritical stationary points in the projective plane.

In this paper, the focus will be on properties of stationary points at infinity and their application to algebraic invariant curves and algebraic integrating factors. We first discuss (in Section 2) nondegenerate stationary points of local analytic (or formal power series) vector fields, using the Poincaré-Dulac normal form. A straightforward application of these results shows that if a vector field f of degree m admits an irreducible invariant algebraic curve, and if all the stationary points of f at infinity are non-degenerate and non-dicritical, then the degree of the curve cannot exceed $m+1$. While this result is also a direct consequence of Carnicer's theorem [5], the proof presented here is elementary and straightforward. Combining the results on stationary points with an investigation of the behavior of integrating factors under Poincaré transforms, we then show that the same conditions on f imply that an algebraic integrating factor is necessarily of the form ϕ^{-1}, where ϕ is a polynomial of degree $m+1$ whose highest-degree term is uniquely (up to a multiplicative constant) determined by f.

The hypotheses on f involve only the homogeneous term $f^{(m)}$ of highest degree. We investigate the relation between properties of $f^{(m)}$ and properties of stationary points at infinity, and in particular we show that the hypotheses are satisfied for almost all (in the Lebesgue sense) homogeneous polynomial vector fields of fixed degree. We also present a criterion that works for vector fields with rational coefficients, which implies that the hypotheses are also, in general, satisfied for such vector fields. Finally, we discuss quadratic vector fields in some detail, and present some examples to illustrate that there are also vector fields with dicritical stationary points at infinity which are accessible by the strategies developed in this paper.

Let us introduce some notation and review a few known facts. (References to be consulted here are Lie [13], Olver [18, 19], Prelle and Singer [21], and Schlomiuk [24, 25, 26], among others.) All functions and vector fields under consideration will be analytic, unless specified otherwise.

To a given differential equation $\dot{x}=f(x)$ on an open, nonempty subset U of $\mathbf{C}^2$, one associates the derivation L_f, assigning to a function ϕ: $U \to \mathbf{C}$ its *Lie derivative* $L_f(\phi)$, with $L_f(\phi)(x) := D\phi(x)\, f(x)$. A (nonconstant) function ψ is called a *first integral*, resp. a *semi-invariant*, of $\dot{x}=f(x)$ (or, briefly, of f) if $L_f(\psi)=0$, resp. $L_f(\psi)=\mu\psi$ for some μ analytic on U. The set of zeros of a semi-invariant, as well as any level set of a first integral, is an invariant set for $\dot{x}=f(x)$.

The commutator of two derivations L_f, L_g is again a derivation, and it turns out to be equal to $L_{[f,g]}$, with the *Lie bracket* $[f, g]$ defined by $[f, g](x)=Dg(x)\,f(x)-Df(x)\,g(x)$. In case $[g, f]=\lambda f$ for some analytic λ, the local one-parameter group of transformations generated by g consists of orbital symmetries of $\dot{x}=f(x)$. This was first noted by Lie [13], and it may be seen as the starting point for the employment of (local) Lie transformation groups in the investigation of differential equations.

We call the vector field f *divergence-free* if $\operatorname{div}(f):=\operatorname{tr} Df=0$. (This notion depends on the choice of coordinates.) To a scalar-valued function ϕ one assigns its *Hamiltonian vector field* $q_\phi:=\begin{pmatrix}-\partial\phi/\partial x_2\\ \partial\phi/\partial x_1\end{pmatrix}$, which is obviously divergence-free, and has first integral ϕ by construction. Locally, a divergence-free vector field is equal to the Hamiltonian vector field of some function. In this sense, finding a first integral of a divergence-free vector field is just an integration problem. It is elementary to verify that $\operatorname{div}(\psi f)=L_f(\psi)+\operatorname{div}(f)\cdot\psi$, so ψf will be divergence-free if and only if $L_f(\psi)+\operatorname{div}(f)\cdot\psi=0$. If ψ is nonzero and this equation is satisfied then we call ψ an *integrating factor* of $\dot{x}=f(x)$. Another function $\tilde{\psi}$ is an integrating factor if and only if $\tilde{\psi}/\psi$ is a first integral.

In dimension two, there is a close relation between integrating factors and generators for local one-parameter symmetry groups, as was already noted by Lie (see Olver [18], Thm. 2.48, Bluman and Cole [2], and [30]): If g is a vector field such that $[g, f]=\lambda f$ for some scalar-valued λ then $1/\det(g(x), f(x))$ is an integrating factor for f (provided that the denominator is not identically zero), and conversely one can construct a nontrivial "infinitesimal symmetry" of f from an integrating factor. This is one way to deduce the behavior of integrating factors under coordinate transformations (cf. [30]): One has to multiply the transformed integrating factor by the Jacobian of the transformation.

While it is known that an integrating factor for $\dot{x}=f(x)$ exists near every nonstationary point (this follows from the fact that the vector field can be straightened near such a point), it is a nontrivial problem to determine such a factor explicitly "in closed form" for a given equation. (There is also the question of existence near a stationary point of the equation.) This is reflected in the fact that finding Lie symmetries of first-order ordinary differential equations is a matter of trial and error, while (for example) the same problem for higher-order equations is amenable to a systematic approach.

We will consider some aspects of the following, more restricted problem: Start with f belonging to a specified class of functions, and consider the problem whether there is an integrating factor in another (possibly larger) class. For a large portion of this article, f will be a polynomial and the search will be for integrating factors which are algebraic over the field $\mathbf{C}(x_1, x_2)$ of rational functions. For the record, we state the following well-known result.

THEOREM 1.1 (Prelle and Singer [21]). *Let $\dot{x} = f(x)$ be given, with f a polynomial.*

(a) *If this equation has a first integral that is elementary (in the sense of differential algebra) over $\mathbf{C}(x_1, x_2)$ then it admits an algebraic integrating factor.*

(b) *If the equation admits an algebraic integrating factor then it also admits an integrating factor of the specific form*

$$(\varphi_1^{d_1} \cdots \varphi_r^{d_r})^{-1},$$

with the φ_i irreducible polynomials (and semi-invariants), and rational numbers d_i. (We include the possibility $r = 0$, with integrating factor 1.)

Actually, part (a) is a consequence of the main result in [21], which states that an elementary first integral must have a quite special form. Since the φ_i in part (b) are semi-invariant polynomials, one has $L_f(\varphi_i) = \mu_i \varphi_i$ with polynomials μ_i, and the integrating factor condition translates into

$$d_1\mu_1 + \cdots + d_r\mu_r = \operatorname{div}(f).$$

Since products and divisors of semi-invariants are again semi-invariant, it is sufficient to consider irreducible ones. From a theorem of Darboux (see Jouanolou [11], Schlomiuk [26]) it is known that every polynomial vector field f admits an upper bound for the possible degrees of irreducible polynomial semi-invariants. If such a bound is known, then deciding the question of existence of algebraic invariant curves and algebraic integrating factors, and finding them in case of an affirmative answer, is a matter of linear algebra. As will be shown, the investigation of stationary points at infinity provides efficient tools to determine such bounds (and possible invariant curves and integrating factors) for a large (generic) class of vector fields, and is also useful in non-generic cases. We will first give an account of local integrating factors of analytic and formal vector fields.

2. THE LOCAL PICTURE

In this section we will investigate invariant sets and integrating factors of the analytic differential equation $\dot{x} = f(x)$ in the neighborhood of a point in $\mathbf{C}^2$, which we may take to be 0. This is interesting in its own right, and we will also be able to use the results obtained here for polynomial differential equations with algebraic integrating factors. The (germs of) functions analytic in 0 form an algebra that is isomorphic to the algebra $\mathbf{C}[[x_1, x_2]]_c$ of power series with a nonempty (open) domain of convergence. It is known that this algebra has a number of nice properties (see, for instance, Ruiz [22]): It is a Noetherian unique factorization domain, and one has a Nullstellensatz (due to Rückert) analogous to Hilbert's Nullstellensatz for polynomials. The invertible elements are those series with a nonzero constant term. Every invertible series can be represented as $\exp(\mu)$, for a suitable series μ. We denote the quotient field of $\mathbf{C}[[x_1, x_2]]_c$ by $\mathbf{C}((x_1, x_2))_c$, and will occasionally refer to its elements as local meromorphic functions. If there are convergence problems, or when convergence issues are irrelevant, we will also deal with the formal counterparts $\mathbf{C}[[x_1, x_2]]$ and $\mathbf{C}((x_1, x_2))$ of these objects, and formal power series vector fields.

Let us introduce the notion of a semi-invariant in this context: A non-invertible $\phi \in \mathbf{C}[[x_1, x_2]]_c$ (thus $\phi(0) = 0$) is called a semi-invariant of f if $L_f(\phi) = \lambda\phi$ for some $\lambda \in \mathbf{C}[[x_1, x_2]]_c$; and the same definition, mutatis mutandis, applies to formal series. (Every invertible series obviously satisfies the defining condition for a semi-invariant, but recall that the interesting feature of an analytic semi-invariant is its set of zeros.) If ϕ is a semi-invariant, with $L_f(\phi) = \lambda\phi$, then so is $\phi \exp(\mu)$ (μ arbitrary), and $L_f(\phi \exp(\mu)) = (\lambda + L_f(\mu))\, \phi \exp(\mu)$.

The results of Prelle and Singer [21] (cf. (1.1)) remain valid in this situation, mutatis mutandis; in fact, they are presented in a general context in [21].

Let us briefly consider the case that 0 is a nonstationary point of f. Then there exists (up to multiplication with invertible series) one and only one irreducible semi-invariant ϕ, and L_f is onto. Both assertions are obvious for the special case $f = \binom{1}{0}$; the general case follows by using the straightening theorem (Olver [18], Prop. 1.29, for instance). It follows that for any rational number d there is an integrating factor $\phi^{-d} \exp(\mu)$, with μ chosen such that the integrating factor condition holds. Therefore, the local situation becomes interesting only at stationary points of f.

Thus we assume that $f(0) = 0$, and define $B := Df(0)$. A crucial technical tool will be a local change of coordinates to simplify the equation. The behavior under coordinate transformations (cf. [30]) shows that an integrating factor $\phi_1^{d_1} \cdots \phi_r^{d_r} \exp(\mu)$ will be changed to the integrating factor $(\phi_1 \circ \Psi)^{d_1} \cdots (\phi_r \circ \Psi)^{d_r} \exp(\nu)$ by an invertible solution-preserving map Ψ, with $\exp(\nu)$ also incorporating the functional determinant of Ψ. Let $B = B_s + B_n$ be the decomposition into semisimple and nilpotent part, and α_1, α_2 the eigenvalues of B. Let us call the stationary point 0 of the differential equation *nondegenerate* if $B_s \neq 0$ (equivalently, not both α_i are zero), and *degenerate* otherwise. For nondegenerate stationary points we have the Poincarè-Dulac normal form at our disposition (see Bibikov [1], Bruno [3], and [28]): There is an invertible formal power series $\Psi = \mathrm{id} + \cdots$ that is solution-preserving from $\dot{x} = f(x)$ to $\dot{x} = \tilde{f}(x)$, with $\tilde{f} = B + \cdots$ a formal power series vector field in normal form, thus satisfying $[B_s, \tilde{f}] = 0$. We list the possible cases in dimension 2; see Bruno [3], for instance.

LEMMA 2.1. *Let* $B_s = \mathrm{diag}(\alpha_1, \alpha_2)$, *with* $\alpha_1 \neq 0$. *Let* $\tilde{f}$ *be in normal form. Then* $\tilde{f}$ *is as follows.*

(a) *If* α_2/α_1 *is not a rational number then* $\tilde{f} = B = B_s$.

(b) *If* $\alpha_2/\alpha_1 = q$, *with* q *a positive integer, then* $\tilde{f} = B_s + \beta\binom{0}{x_1^q}$ *for some* $\beta \in \mathbf{C}$.

(c) *If* $\alpha_2/\alpha_1 = q/p$, *with* p *and* q *relatively prime positive integers,* $1 < p < q$, *then* $B = B_s$ *and* $\tilde{f} = B_s$.

(d) *If* $\alpha_2/\alpha_1 = -q/p$, *with* p *and* q *relatively prime positive integers,* $p \leqslant q$, *then* $B = B_s$ *and* $\tilde{f} = B + \sum_{j \geqslant 1} \gamma^j(\sigma_j\, \mathrm{id} + \tau_j B)$, *where* $\sigma_j, \tau_j \in \mathbf{C}$, *and* $\gamma(x) := x_1^q x_2^p$.

(e) *If* $\alpha_2 = 0$ *then* $B = B_s$ *and* $\tilde{f} = B + \sum_{j \geqslant 1} \gamma^j(\sigma_j\, \mathrm{id} + \tau_j B)$, *where* $\sigma_j, \tau_j \in \mathbf{C}$, *and* $\gamma(x) := x_2$.

In cases (d) and (e), γ is a first integral of B, and every first integral of B in $\mathbf{C}[[x_1, x_2]]$ is a series in powers of γ.

As noted before, there is always a formal power series transformation from $\dot{x} = f(x)$ to $\dot{x} = \tilde{f}(x)$. For analytic f, there may be convergence problems in case (a) (due to small denominators), and in cases (d) and (e). In the latter cases, existence of a convergent transformation is guaranteed when all $\sigma_j = 0$ (this is Bruno's "Condition A"; cf. [3]). Bibikov [1] investigated the existence of smooth analytic invariant manifolds tangent to the eigenspaces of B_s. His results imply that there exist such manifolds in cases (a), (c), (d) and (e). In case (b), with $\beta \neq 0$, there will only be one such manifold (tangent to $x_1 = 0$), while there are infinitely many when $\beta = 0$. It should also be mentioned that Martinet and Ramis [15, 16] gave a classification up to analytical equivalence of cases (d) and (e).

Differential equations in normal form admit integrating factors which can be obtained directly from Lie's result mentioned in the Introduction: Take $\det(B_s, \tilde{f})^{-1}$, unless the determinant is identically zero; otherwise $(x_1 x_2)^{-1}$ will work.

We will give a detailed enumeration of irreducible semi-invariants, algebraic first integrals, and algebraic integrating factors for all of the cases (a) through (e) above. These results are essentially known; note, for instance, that some of them are mentioned in Mattei/Moussu [17], and

Martinet/Ramis [16]. It seems, though, that a short, complete account including the formal case is not easy to find. We start with a result on semi-invariants (thus, on invariant sets of codimension one) of differential equations in normal form. Since this is of interest in arbitrary dimension, and the proof in the general case is not much more difficult, we state this for formal vector fields on $\mathbf{C}^d$, with arbitrary d. (The result generalizes Prop. 1.8 of [28], and the proof follows similar lines.)

LEMMA 2.2. *Let B be linear on $\mathbf{C}^d$, with decomposition $B = B_s + B_n$ into semisimple and nilpotent part. Let $f^{(j)}$ be a polynomial vector field on $\mathbf{C}^d$ that is homogeneous of degree j ($j \geqslant 2$), and suppose that the (formal) vector field $f := B + \sum f^{(j)}$ is in normal form, thus $[B_s, f] = 0$.*

If $\phi = \sum_{j \geqslant r} \phi_j$ (with $\phi_r \neq 0$) and $\lambda = \sum_{i \geqslant 0} \lambda_i$ satisfy $L_f(\phi) = \lambda\phi$ (thus ϕ is a formal semi-invariant of f), then there is an invertible formal series β such that $\phi^* := \beta\phi = \sum_{j \geqslant r} \phi_r^*$ satisfies $L_f(\phi^*) = \lambda^*\phi^*$, with $L_{B_s}(\lambda^*) = 0$. Moreover, $L_{B_s}(\phi^*) = \lambda_0 \phi^*$.

Proof. Semi-invariance of ϕ is equivalent to

$$L_B(\phi_{r+j}) + L_{f^{(2)}}(\phi_{r+j-1}) + \cdots L_{f^{(j+1)}}(\phi_r) = \lambda_0\phi_{r+j} + \lambda_1\phi_{r+j-1} + \cdots \lambda_j\phi_r$$

for all $r \geqslant 0$.

Note that $L_{B_s}(\lambda_0) = 0$. Now assume that $L_{B_s}(\lambda_j) = 0$ for all $j < k$, and let $\tilde{\phi} := (1 + \beta_k)\,\phi$, with β_k some form of degree k. Then $\tilde{\phi} = \phi_r + \cdots + \phi_{r+k-1} + (\phi_{r+k} + \beta_k\phi_r) + \cdots$, and $L_f(\tilde{\phi}) = (1 + \beta_k)\,L_f(\phi) + \phi L_f(\beta_k) = \tilde{\lambda}\tilde{\phi}$, with $\tilde{\lambda} = \lambda_0 + \cdots \lambda_{k-1} + (\lambda_k + L_B(\beta_k) + \cdots$. Now L_{B_s} acts semisimple on the space $\mathscr{S}_k$ of all forms of degree k, which is therefore the direct sum of kernel and image of this map. Since the restriction of L_B to the image is invertible, it is possible to choose β_k in such a way that $L_{B_s}(\lambda_k + L_B(\beta_k)) = 0$. The first assertion now follows by induction. (Note that a product $(1 + \sigma_1)(1 + \sigma_2)\cdots$, with the σ_i homogeneous of degree i, is convergent in the setting of formal power series!)

As to the second assertion, assume that $L_{B_s}(\phi^*_{r+j}) = \lambda_0\phi^*_{r+j}$ for all $j < k$. Since L_B and all the $L_{f^{(i)}}$ commute with L_{B_s}, one has

$$\begin{aligned} &L_B(\phi^*_{r+k}) - \lambda_0\phi^*_{r+k} \\ &\qquad = -(L_{f^{(2)}}(\phi^*_{r+k-1}) + \cdots + L_{f^{(k+1)}}(\phi^*_r)) + \lambda_1^*\phi^*_{r+k-1} + \cdots \lambda_k^*\phi_r^* =: \psi, \end{aligned}$$

with $L_{B_s}(\psi) = \lambda_0\psi$. From semisimplicity of L_{B_s} it follows that $L_{B_s}(\phi^*_{r+k}) = \lambda_0\phi^*_{r+k}$. ∎

We use this in dimension 2:

THEOREM 2.3. *Let $B_s = \mathrm{diag}(\alpha_1, \alpha_2)$, with $\alpha_1 \neq 0$. Let f be a formal power series vector field in normal form.*

(a) *If α_2/α_1 is not a rational number then x_1 and x_2 are (up to multiplication with invertible series) the only irreducible semi-invariants of f in $\mathbf{C}[[x_1, x_2]]$. There is no first integral that is algebraic over $\mathbf{C}((x_1, x_2))$, and the only integrating factor algebraic over $\mathbf{C}((x_1, x_2))$, up to multiplication by scalars, is $(x_1x_2)^{-1}$.*

(b) *Let $\alpha_2/\alpha_1 = q$, with q a positive integer, and $f = B_s + \beta\binom{0}{x_1^q}$, with $\beta \in \mathbf{C}$.*

If $\beta \neq 0$ then x_1 is (up to multiplication with invertible series) the only irreducible semi-invariant of f in $\mathbf{C}[[x_1, x_2]]$. There is no first integral algebraic over $\mathbf{C}((x_1, x_2))$, and the only integrating factor algebraic over $\mathbf{C}((x_1, x_2))$, up to multiplication by scalars, is $x_1^{-(q+1)}$.

If $\beta = 0$ then the irreducible semi-invariants (up to multiplication with invertible series) are x_1 and all $x_2 + \delta x_1^q$, with $\delta \in \mathbf{C}$. Every product $(x_1^e \prod (x_2 + \delta_i x_1^q)^{d_i})^{-1}$ (with rational exponents) is a first integral if and only if $e + q\sum d_i = 0$, and an integrating factor if and only if $e + q\sum d_i = 1 + q$. Moreover, up to multiplication by constants, these are the only algebraic first integrals resp. integrating factors that are products of powers of semi-invariants and invertible series.

(c) *Let $\alpha_2/\alpha_1 = q/p$ as in (2.1c).*

Then the irreducible semi-invariants (up to multiplication with invertible series) are x_1, x_2 and all $x_2^p + \delta x_1^q$, with $0 \neq \delta \in \mathbf{C}$. Every product $(x_1^{e_1} x_2^{e_2} \prod (x_2^p + \delta_i x_1^q)^{d_i})^{-1}$ (with rational exponents) is a first integral if and only if $pe_1 + qe_2 + pq\sum d_i = 0$, and an integrating factor if and only if $pe_1 + qe_2 + pq\sum d_i = p + q$. Moreover, up to multiplication by constants, these are the only algebraic first integrals resp. integrating factors that are products of powers of semi-invariants and invertible series.

(d) *Let $\alpha_2/\alpha_1 = -q/p$, with p and q relatively prime positive integers, $p \leqslant q$, and $f = B + \sum_{j \geqslant 1} \gamma^j(\sigma_j \,\mathrm{id} + \tau_j B)$, where $\sigma_j, \tau_j \in \mathbf{C}$, and $\gamma(x) := x_1^q x_2^p$. Then x_1 and x_2 are, up to multiplication with invertible series, the only irreducible semi-invariants of f in $\mathbf{C}[[x_1, x_2]]$.*

If not all the σ_j are equal to zero then there exists no first integral that is algebraic over $\mathbf{C}((x_1, x_2))$. Up to multiplication by constants, the only integrating factor that is algebraic over $\mathbf{C}((x_1, x_2))$ is $(x_1^{1+lq} x_2^{1+lp} \rho(x))^{-1} = (x_1 x_2 \gamma(x)^l \rho(x))^{-1}$, with l the smallest index such that $\sigma_l \neq 0$, and $x_1 x_2 \rho(x) = \det(B(x), \sigma_l x + \sum \gamma(x)^i \sigma_{l+i}(x))$.

If all $\sigma_j = 0$ then $\gamma(x)^d \cdot \rho(\gamma(x))$ is a first integral algebraic over $\mathbf{C}((x_1, x_2))$ for any rational d and any invertible formal power series ρ in one variable, and $(x_1 x_2 \gamma(x)^d \cdot \rho(\gamma(x)))^{-1}$ is an integrating factor. Moreover, every *first integral that is a product of powers of semi-invariants, multiplied by some invertible series, is of this type.*

(e) *Let* $\alpha_2 = 0$, *and* $f = B + \sum_{j \geq 1} \gamma^j(\sigma_j\, \mathrm{id} + \tau_j B)$, *where* $\sigma_j, \tau_j \in \mathbf{C}$, *and* $\gamma(x) := x_2$. *Then* x_1 *and* x_2 *are, up to multiplication with invertible series, the only irreducible semi-invariants of* f *in* $\mathbf{C}[[x_1, x_2]]$.

If not all the σ_j *are equal to zero then there exists no first integral that is algebraic over* $\mathbf{C}((x_1, x_2))$. *Up to multiplication by constants, the only integrating factor that is algebraic over* $\mathbf{C}((x_1, x_2))$ *is* $(x_1 x_2^{1+l}\rho(x))^{-1} = (x_1 x_2 \gamma(x)^l \rho(x))^{-1}$, *with* l *the smallest index such that* $\sigma_l \neq 0$, *and* ρ *is defined similarly to* (d).

If all $\sigma_j = 0$ *then* $\gamma(x)^d \cdot \rho(\gamma(x))$ *is a first integral algebraic over* $\mathbf{C}((x_1, x_2))$ *for any rational* d *and any invertible formal power series* ρ *in one variable, and* $(x_1 x_2 \gamma(x)^d \cdot \rho(\gamma(x))^{-1}$ *is an integrating factor. Moreover, every first integral that is a product of powers of semi-invariants, multiplied by some invertible series, is of this type.*

Proof. (i) Let ϕ be a semi-invariant of f. It was shown in Lemma 2.2 that we may assume $L_{B_s}(\phi) = \lambda_0 \phi$, with some $\lambda_0 \in \mathbf{C}$. From $L_{B_s}(x_1^{i_1} x_2^{i_2}) = (i_1\alpha_1 + i_2\alpha_2)(x_1^{i_1} x_2^{i_2})$ it follows that $\lambda_0 = (k_1\alpha_1 + k_2\alpha_2)$ for certain non-negative k_1, k_2. If $(j_1, j_2) \neq (k_1, k_2)$ and $k_1\alpha_1 + k_2\alpha_2 = j_1\alpha_1 + j_2\alpha_2$ then α_2/α_1 is rational. Therefore, in case (a) the only semi-invariants are the $x_1^{i_1} x_2^{i_2}$, and only x_1 and x_2 are irreducible.

In cases (b) and (c) (with $p := 1$ in case (b)) we may assume that $\alpha_1 = p$ and $\alpha_2 = q$. Let $\lambda_0 = k_1 p + k_2 q$. Then $L_{B_s}(x_1^{i_1} x_2^{i_2}) = \lambda_0 x_1^{i_1} x_2^{i_2}$ if and only if $p(k_1 - i_1) = q(i_2 - k_2)$, thus p divides $i_2 - k_2$, and q divides $k_1 - i_1$. It follows that $L_{B_s}(\phi) = \lambda_0 \phi$ if and only if $\phi(x) = x_1^{e_1} x_2^{e_2} \cdot \psi(x_1^q, x_2^p)$ for suitable e_1, e_2, and a homogeneous polynomial ψ in two variables. Since ψ is a product of linear polynomials, the assertion on semi-invariants follows in case (c), and in (b) for $\beta = 0$. If $\beta \neq 0$ in case (b), the assertion follows immediately from (2.2).

In case (d) we may assume that $\alpha_1 = p$ and $\alpha_2 = -q$. Let $\lambda_0 = k_1 p - k_2 q$. Then $L_{B_s}(x_1^{i_1} x_2^{i_2}) = \lambda_0 x_1^{i_1} x_2^{i_2}$ if and only if $p(k_1 - i_1) = q(k_2 - i_2)$, thus p divides $k_2 - i_2$, and q divides $k_1 - i_1$. It follows that $L_{B_s}(\phi) = \lambda_0 \phi$ if and only if $\phi = x_1^{i_1} x_2^{i_2} \cdot \rho(x_1^q x_2^p)$, with ρ an invertible series in one variable. The assertion about semi-invariants follows. In case (e) similar arguments apply.

(ii) It is elementary to verify that $(x_1 x_2)^{-1}$ is an integrating factor whenever $f = B_s$. In the other cases, use Lie's theorem to obtain the integrating factor $\det(B(x), f(x))^{-1}$. This works (and yields the integrating factors as asserted) unless all $\sigma_j = 0$ in cases (d) or (e). But then $f(x) = (1 + \sum \tau_j \gamma(x)^j)\, B(x)$, and $((1 + \sum \tau_j \gamma(x)^j)\, x_1 x_2)^{-1}$ is obviously an integrating factor. In each case, other integrating factors (should they exist) differ from the ones determined here by multiplication with a first integral.

(iii) It can be directly verified that the functions given in parts (b) through (d) (if applicable) are first integrals resp. integrating factors, and it only remains to be seen that there are no others. Now, the existence of a (nonconstant) first integral that is algebraic over $\mathbf{C}((x_1, x_2))$ implies the existence of a first integral in $\mathbf{C}((x_1, x_2))$ itself. (All the coefficients in the minimum polynomial of such a first integral are annihilated by L_f, and not all of them are constant; see also Prelle and Singer [21].) Thus we may assume that $\psi = \psi_1^{m_1} \cdots \psi_r^{m_r} \exp(\mu)$ is a first integral, with $r \geqslant 0$, semi-invariants ψ_j, and integers m_j. We note that $L_f(\exp(\mu)) = L_f(\mu) \exp(\mu)$, and $L_f(\mu)(0) = 0$.

In case (a), we have $\psi = x_1^{m_1} x_2^{m_2} \exp(\mu)$, and $L_B(\psi) = 0$ if and only if $m_1 \alpha_1 + m_2 \alpha_2 + L_B(\mu) = 0$. Since α_2/α_1 is not rational, and $L_B(\mu)(0) = 0$, this implies $m_1 = m_2 = 0$, and $L_B(\mu) = 0$. As was seen in (i), μ is then constant.

In cases (b), when $\beta = 0$, and (c), we have $\psi = x_1^{m_1} x_2^{m_2} \prod (x_1^q + \delta_i x_2^p)^{n_i}$ $\exp(\mu)$, and analogous reasoning shows that $pm_1 + qm_2 + pq \sum n_i = 0$, and that μ is constant. In case (b), with $\beta \neq 0$, the same argument works with $\psi = x_1^{m_1} \exp(\mu)$.

In cases (d) and (e), according to (i) and Lemma 2.2, we may assume that $\psi = \frac{\psi_1}{\psi_2} \exp(\mu)$, with $L_B(\psi_j) = \lambda_0 \psi_j$, and $\lambda_0 \in \mathbf{C}$, hence $\psi_j = x_1^{k_1} x_2^{k_2} \cdot v_j(\gamma(x))$, with series v_j in one variable. We may assume that the order of v_2 is less than or equal to that of v_1, which then shows that $\psi \in \mathbf{C}[[x_1, x_2]]$. Now Lemma 2.2 shows that ψ is a series in γ, and γ is a first integral of f. From $L_f(\gamma) = \operatorname{div}(B) \sum \sigma_j \gamma^{j+1}$ the assertions follow. ∎

It is remarkable that one gets very precise information in every case except when α_2/α_1 is rational and positive. We will refer to the corresponding stationary points as "rational nodes" (adopting terminology from the classification of real linear systems). These points, with the exception of the first case in (2.3b) are precisely the dicritical ones among the nondegenerate stationary points, i.e., they admit infinitely many (pairwise relatively prime) semi-invariants. For nondegenerate stationary points which are not dicritical the theorem yields strong consequences.

COROLLARY 2.4. *Let $\dot{x} = f(x)$ be a plane polynomial differential equation, and z a nondegenerate stationary point which is not dicritical. Let $\phi_1, ..., \phi_r$ be irreducible and pairwise relatively prime polynomials.*

(a) *If the ϕ_i are semi-invariant then $\phi_i(z) = 0$ for at most two indices.*

(b) *Let $d_1, ..., d_r$ be rational numbers such that $(\phi_1^{d_1} \cdots \phi_r^{d_r})^{-1}$ is an integrating factor for f.*

(b1) *If $\phi_i(z) \neq 0$ for all i then the eigenvalue ratio is as in* (2.1d, e), *and Bruno's Condition A is satisfied.*

(b2) *Assume that there is an* $s > 0$ *such that* $\phi_1(z) = \cdots = \phi_s(z) = 0$ *and* $\phi_i(z) \neq 0$ *for all* $i > s$. *Then* $s \leqslant 2$. *Moreover*:

— *In the situation of* (2.1a), *one has either* $s = 1$ *and* $d_1 = 1$ *or* $s = 2$ *and* $d_1 = d_2 = 1$.

— *In the situation of* (2.1b), *with* $\beta \neq 0$, *one has* $s = 1$ *and* $d_1 = q + 1$.

— *In the situation of* (2.1d), *with* $p \neq q$, *one has* $s = 1$ *only if Bruno's Condition A is satisfied, and then necessarily* $d_1 = 1$. *If* $s = 2$ *then* (*up to a permutation*) $d_1 = 1 + lq$, $d_2 = 1 + lp$ *for some rational number* l. *If Bruno's Condition A is not satisfied then* l *is an integer and* $l > 0$.

— *In the situation of* (2.1d), *with* $p = q = 1$, *one has either* $s = 1$ *and* $d_1 = 1 + l$ *or* $s = 2$ *and* $d_1 = d_2 = 1 + l$ *for some rational number* l. *If Bruno's Condition A is not satisfied then* l *is an integer and* $l > 0$.

— *In the situation of* (2.1e) *one has* $s = 1$ *only if the stationary point* z *is not isolated, and then necessarily* $d_1 = 1$. *If* $s = 2$ *then* $d_1 = 1$, $d_2 = 1 + l$ *for some rational number* l. *If* z *is an isolated stationary point then* l *is an integer and* $l > 0$.

Proof. Whenever $\phi_i(z) = 0$, there is a prime factorization of ϕ_i in the formal power series ring corresponding to z. Now use the results of (2.3). Note that in the last case Bruno's Condition A is satisfied if and only if the stationary point z is not isolated. ▮

There is another consequence of (2.3) that deserves particular attention.

Remark 2.5. Let f be a real analytic vector field in a neighborhood of the stationary point $z \in \mathbf{R}^2$. If $Df(z)$ has nonzero purely imaginary eigenvalues, and $(\phi_1^{d_1} \cdots \phi_r^{d_r})^{-1}$ is an integrating factor of f, with all ϕ_i analytic in z, then z is a center unless $\phi_i(z) = 0$ for some i and $d_i = 1 + l$ with some integer $l > 0$.

In particular, if f is a polynomial vector field with integrating factor $(\phi_1 \cdots \phi_r)^{-1}$, with polynomials ϕ_i, then every stationary point z whose linearization yields a center is already a center.

Proof. From (2.3) one sees that Bruno's Condition A must be satisfied if no ϕ_i vanishes at z or $\phi_j(z) = 0$ with $d_i \neq 1 + k$ for all positive integers k, and this in turn guarantees convergence of a normalizing transformation and existence of a nonconstant analytic first integral in z. ▮

The special situation described in the second part of (2.5) is encountered quite frequently, provided that some algebraic integrating factor exists; see Kooij and Christopher [12], and Section 3 of this article.

As should be expected, the results of (2.3) also yield criteria for non-existence of algebraic integrating factors. This will not be discussed here in detail, but one example should be presented.

EXAMPLE 2.6. The equation

$$\dot{x}=\begin{pmatrix} x_1+3x_2+x_1^2 \\ 3x_1+x_2+\frac{9}{2}x_1^2+\frac{3}{2}x_1x_2 \end{pmatrix}$$

does not admit an algebraic integrating factor.

Proof. By construction, $\phi_1(x)=x_1^2+x_1^3-x_2^2$ is a semi-invariant of this equation. Furthermore, the linearization of the equation at the stationary point 0 is given by $\left(\begin{smallmatrix}1&3\\3&1\end{smallmatrix}\right)$, with eigenvalues 4 and -2.

Locally, $\phi_1=(x_2+x_1\sqrt{1+x_1})(x_2-x_1\sqrt{1+x_1})=\tilde{x}_1\tilde{x}_2$ is a product of two irreducible factors. Now (2.4) shows that, if there exists an integrating factor $\gamma=(\phi_1^{d_1}\cdots\phi_r^{d_r})^{-1}$ that is algebraic over $\mathbf{C}(x_1, x_2)$, then all the ϕ_i with $i>1$ are invertible at 0, and locally one must have $\gamma=(\tilde{x}_1^{d_1}\tilde{x}_2^{d_1}\exp(\mu))^{-1}$, with equal exponents for $\tilde{x}_1$ and $\tilde{x}_2$. On the other hand, computing the normal form up to degree 4 (with $\alpha_2/\alpha_1=-1/2$ in the terminology of (2.1)ff.) yields $\sigma_1\neq 0$, and then, according to (2.3) and (2.4), a unique integrating factor $(\tilde{x}_1^2\tilde{x}_2^3\exp(\nu))^{-1}$, and this is a contradiction.

3. APPLICATIONS TO POLYNOMIAL SYSTEMS

From now on, we consider an equation $\dot{x}=f(x)$ in $\mathbf{C}^2$, with f a polynomial. The local theory developed in the previous section yields conditions on the possible singular points of semi-invariants, and the possible exponents occurring in integrating factors. Note that the common points of two different invariant (algebraic) curves are stationary, and that the number of such points is determined by the degrees of the defining poynomials. For a proper discussion of this one has to pass to the projective plane. Recall *Bezout's theorem* in the projective plane $\mathbf{P}^2$ over $\mathbf{C}$ (see, for instance, Shafarevich [27]):

Two homogeneous, relatively prime polynomials $\tilde{\rho}(x_1, x_2, x_3)$ and $\tilde{\sigma}(x_1, x_2, x_3)$, of respective degrees $r>0$ and $s>0$, have exactly $r\cdot s$ common zeros in $\mathbf{P}^2$, with multiplicities counted properly. For relatively prime polynomials $\rho(x_1, x_2)=\sum_{i=0}^r \rho_i(x_1, x_2)$, and $\sigma(x_1, x_2)=\sum_{i=0}^s \sigma_i(x_1, x_2)$ in $\mathbf{C}[x_1, x_2]$ (with $r>0$, $s>0$, the ρ_i and σ_i homogeneous of degree i, and $\rho_r\neq 0$, $\sigma_s\neq 0$), Bezout's theorem is applicable to the *homogenizations* $\tilde{\rho}:=\sum_{i=0}^r \rho_i(x_1, x_2)\,x_3^{r-i}$, resp. $\tilde{\sigma}:=\sum_{i=0}^s \sigma_i(x_1, x_2)\,x_3^{s-i}$.

Common zeros of ρ and σ are in 1–1-correspondence with those common zeros $(y_1 : y_2 : y_3)$ of $\tilde{\rho}$ and $\tilde{\sigma}$ that satisfy $y_3 \neq 0$. Thus the polynomials ρ and σ have at most $r \cdot s$ common zeros in $\mathbf{C}^2$. (If ρ and σ have a common zero, say, at 0, then its multiplicity is the (finite) dimension of the vector space $\mathbf{C}[[x_1, x_2]]/(\rho, \sigma)$. In particular, this multiplicity is equal to 1 if and only if the derivatives $D\rho(0)$ and $D\sigma(0)$ are linearly independent.) The common zeros $(y_1 : y_2 : y_3)$ of $\tilde{\rho}$ and $\tilde{\sigma}$ with $y_3 = 0$ are in 1–1-correspondence with the common factors of ρ_r and σ_s (with multiplicities corresponding).

Given $\dot{x} = f(x)$, with the entries of f of degree m (or smaller), and relatively prime, it follows that there are at most m^2 stationary points. Assume that

$$\gamma := (\phi_1^{m_1/q} \cdots \phi_r^{m_r/q})^{-1} \tag{$\dagger$}$$

is an integrating factor, with the ϕ_i irreducible (and pairwise relatively prime) polynomials of degree n_i, respectively, and relatively prime integers $m_1, ..., m_r, q$.

Since every common zero of ϕ_i and ϕ_j (with $i \neq j$) is a stationary point of $\dot{x} = f(x)$, and, counting multiplicities, the homogenized polynomials have $n_i \cdot n_j$ common zeros in the projective plane, this raises some hope for degree bounds. To effectively find such bounds, one still needs to take stationary points at infinity into account. The geometric background of this procedure (going back to Poincaré) is a correspondence between plane polynomial vector fields and certain polynomial vector fields on a sphere (covering $\mathbf{P}^2$), such that an equator of the sphere can be identified with points at infinity of the plane. (See Perko [20], Section 3.10, for the procedure, and also Schlomiuk [24]. The algebraic version was investigated in [23]; in particular it works equally well in the complex case.) For our purpose, the geometric interpretation is of little importance, and the necessary ingredients will be presented directly.

We introduce some notation. For a polynomial $\phi(x_1, x_2) = \sum_{i=0}^r \phi_i(x_1, x_2) \in \mathbf{C}[x_1, x_2]$, of degree r, let $\tilde{\phi}(x_1, x_2, x_3) = \sum_{i=0}^r \phi_i(x_1, x_2)\, x_3^{r-i}$ be its homogenization, and call $\phi^*(x_2, x_3) := \tilde{\phi}(1, x_2, x_3)$ the *Poincaré transform* of ϕ with respect to x_1. (The points $(y_1 : y_2 : 0)$ of the projective plane are also called points at infinity. The name "Poincaré transform" is not in common use for this situation; it will be used here to have uniform notation for polynomials and vector fields.)

For a polynomial vector field $f(x_1, x_2) = \sum_{k=0}^m f^{(k)}(x_1, x_2)$, with $f^{(k)}$ homogeneous of degree k, and $f^{(m)} \neq 0$, let

$$g(x_1, x_2, x_3) := \begin{pmatrix} \sum\limits_{k=0}^m f^{(k)}(x_1, x_2)\, x_3^{m-k} \\ 0 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ 0 \end{pmatrix}$$

be its homogenization, and furthermore

$$\tilde{g}(x_1, x_2, x_3) := -g_1 \cdot x + x_1 \cdot g = \begin{pmatrix} 0 \\ -x_2 g_1 + x_1 g_2 \\ -x_3 g_1 \end{pmatrix}$$

its projection with respect to x_1 (as it is called in [23]). The *Poincaré transform* of f with respect to the vector $\binom{1}{0}$ is then

$$f^*(x_2, x_3) := \begin{pmatrix} -x_2 g_1(1, x_2, x_3) + g_2(1, x_2, x_3) \\ -x_3 g_1(1, x_2, x_3) \end{pmatrix}.$$

(This is the Poincaré transform as it can be found in Perko [20], for instance. The intermediate versions have also been recorded because they will be needed later on. Note that f^* has the built-in semi-invariant x_3.)

For a nonzero $v = \binom{\beta_1}{\beta_2} \in \mathbf{C}^2$, let T be an invertible linear map satisfying $Tv = \binom{1}{0}$ and define a Poincaré transform ϕ_v^* of ϕ with respect to v as $(\phi \circ T^{-1})^*$, and a Poincaré transform f_v^* of f with respect to v as $(T \circ f \circ T^{-1})^*$. (There are more canonical ways to define these, and there are some arbitrary choices in the definitions, but this is sufficient for our purposes.)

We now collect a few properties of Poincaré transforms, keeping the notation from above.

LEMMA 3.1. (a) *If ϕ is irreducible then ϕ_v^* is irreducible. One has $\phi_v^*(0) = 0$ if and only if $-\beta_2 x_1 + \beta_1 x_2$ divides the highest degree term $\phi^{(r)}$. In that case, the zero sets of ϕ_v^* and x_3 meet transversally at 0 if and only if $(-\beta_2 x_1 + \beta_1 x_2)^2$ does not divide $\phi^{(r)}$.*

(b) *One has $f_v^*(0) = 0$ if and only if the vector v satisfies $f^{(m)}(v) = \lambda v$ for some $\lambda \in \mathbf{C}$. In that case, if the eigenvalues of $Df^{(m)}(v)$ are $m\lambda$ (with eigenvector v, due to homogeneity), and ν, then the eigenvalues of $Df_v^*(0)$ are $\nu - \lambda$ (with eigenspace $x_3 = 0$), and $-\lambda$.*

Proof. It is harmless to assume $v = \binom{1}{0}$.

(a) The assertion about irreducibility follows from a standard result on homogenization and dehomogenization. Let $\phi^{(r)} = \sum_{i=0}^{r} \sigma_i x_1^{r-i} x_2^i$. Then $\phi^*(0) = \sigma_0$, and $D\phi^*(0) = (\sigma_1, *)$. This shows the remaining assertions.

(b) Let

$$f^{(m)}=\begin{pmatrix} \sum_{i=0}^{m} \rho_i x_1^{m-i} x_2^i \\ \sum_{i=0}^{m} \sigma_i x_1^{m-i} x_2^i \end{pmatrix}.$$

Then $f^*(0)=0$ if and only if $g_2(1,0,0)=0$, hence $\sigma_0=0$, and the first assertion follows (with $\lambda=\rho_0$). Note that

$$Df^{(m)}(1,0)=\begin{pmatrix} m\rho_0 & * \\ 0 & \sigma_1 \end{pmatrix},$$

hence $\nu=\sigma_1$. Now the second assertion follows from

$$f^*=\begin{pmatrix} -x_2(\rho_0+\text{t.h.o.})+(\sigma_1 x_2+\text{t.h.o.}) \\ -x_3(\rho_0+\text{t.h.o.}) \end{pmatrix}. \quad \blacksquare$$

Let $0\neq v\in\mathbf{C}^2$ such that $f^{(m)}(v)\in\mathbf{C}v$. Then we call $\mathbf{C}v$ a *stationary point at infinity* of f. (This is justified because $\mathbf{C}v$ is a stationary point at infinity if and only if 0 is a stationary point of the Poincaré transform f_v^*.) Since $f^{(m)}(w)\in\mathbf{C}w$ if and only if w is a zero of the homogeneous polynomial $\rho(x):=\det(x, f^{(m)}(x))$, there are at most $m+1$ stationary points at infinity unless $\rho=0$. Note that $\rho=0$ if and only if $f^{(m)}(x)=\eta(x)\cdot x$ with some $(m-1)$-form η. If $f^{(m)}(v)=\lambda v$, with $\lambda\neq 0$, then we will call the number $\frac{\lambda-\nu}{\lambda}$ the *eigenvalue ratio of the linearization* at the given stationary point at infinity.

The eigenvalue ratios of the stationary points at infinity are not independent, as the next result shows.

Proposition 3.2. *Let f be such that there are exactly $m+1$ stationary points at infinity, and that all of them are nondegenerate, with nonzero eigenvalue ratios $\mu_1, ..., \mu_{m+1}$. Then*

$$\sum_{i=1}^{m+1} \mu_i^{-1}=1.$$

Proof. This is a consequence of a theorem by Camacho and Sad, cf. [4], Appendix, as was shown in Lins Neto [14], Section 1.2, Remarks. (See also the Introduction of Cerveau and Lins Neto [6].) $\blacksquare$

Let us now investigate the behavior of semi-invariants and integrating factors under Poincaré transforms.

PROPOSITION 3.3. *Let $\dot{x} = f(x)$ be given, with m the degree of f.*

(a) *If ϕ is a semi-invariant of f, of degree n, with $L_f(\phi) = \lambda\phi$, then ϕ^* is a semi-invariant of f^*, with $L_{f^*}(\phi^*)(x_2, x_3) = (-ng_1(1, x_2, x_3) + \lambda^*(x_2, x_3))\, \phi^*(x_2, x_3)$.*

(b) *If f admits an integrating factor as in (†) then*

$$(x_3^{(m+2-(m_1 n_1 + \cdots + m_r n_r)/q)} \phi_1^{*(m_1/q)} \cdots \phi_r^{*(m_r/q)})^{-1}$$

is an integrating factor of f^*.

Proof. We use the notation introduced above.

(a) From $L_f(\phi) = \lambda\phi$ it follows that $L_g(\tilde{\phi}) = \tilde{\lambda}\tilde{\phi}$, and then that $L_{\tilde{g}}(\tilde{\phi}) = (-ng_1 + x_1\tilde{\lambda})\, \tilde{\phi}$, and the assertion follows upon setting $x_1 = 1$.

(b) One computes

$$\begin{aligned}
\operatorname{div}(\tilde{g}) &= -\frac{\partial g_1}{\partial x_2} \cdot x_2 - g_1 + \frac{\partial g_2}{\partial x_2} \cdot x_1 - \frac{\partial g_1}{\partial x_3} \cdot x_3 - g_1 \\
&= -2g_1 - \left(\frac{\partial g_1}{\partial x_1} \cdot x_1 + \frac{\partial g_1}{\partial x_2} \cdot x_2 + \frac{\partial g_1}{\partial x_3} \cdot x_3 \right) + x_1 \operatorname{div}(g) \\
&= -(m+2)\, g_1 + x_1\, \widetilde{\operatorname{div}(f)},
\end{aligned}$$

since g_1 is homogeneous of degree m, and $\operatorname{div}(g)$ is, in this special case, the homogenization of $\operatorname{div}(f)$. This yields

$$\operatorname{div}(f^*) = -(m+2)\, g_1(1, x_2, x_3) + (\operatorname{div}(f))^*.$$

With $L_{f^*}(x_3^d) = -d\, g_1(1, x_2, x_3) \cdot x_3^d$ the assertion follows. ∎

Note that these results remain valid for Poincaré transforms with respect to an arbitrary nonzero vector, since linear transformations (with constant Jacobian) change an integrating factor only by a multiplicative constant.

The following result, the first one addressing the Poincaré problem, is not new. It is, for instance, a consequence of a theorem of Carnicer [5], and may also be deduced from Cerveau and Lins Neto [6]. It is included here because we can supply a straightforward and elementary proof.

THEOREM 3.4. *Assume that all the stationary points at infinity of $\dot{x} = f(x)$ are nondegenerate, and that none of them is a rational node. Let $\psi_1, ..., \psi_s$, with $s \leqslant m+1$, be the (pairwise relatively prime) linear forms dividing* $\det(x, f^{(m)}(x))$.

If ϕ is an irreducible semi-invariant of f then the highest-degree term of ϕ is a product $\psi_{l_1} \cdots \psi_{l_d}$ (up to a scalar factor), with pairwise different l_j. None *of the* ψ_{l_j} *divides the highest-degree term of any other irreducible semi-invariant.*

In particular, the sum of the degrees of all the irreducible semi-invariants of f *is at most* $m+1$.

Proof. Every ψ_i determines a straight line $\mathbf{C}\cdot v_i$, which, in turn, determines a stationary point at infinity, and all stationary points at infinity are determined like this. Now let ϕ be a semi-invariant of f. Then the highest-degree term of ϕ is a semi-invariant of $f^{(m)}$, and thus a product $\psi_{l_1}^{k_1}\cdots\psi_{l_d}^{k_d}$, with pairwise different l_j, and positive integers k_j. Consider the Poincaré transform $\hat{\phi}$ with respect to ψ_{l_j}. Since the (invariant) zero sets of $\hat{\phi}$ and x_3 meet at (0, 0), this point is stationary for the corresponding Poincaré transform of f. According to hypothesis, this stationary point at infinity is nondegenerate and not a rational node, and hence the (local) zero sets of the Poincaré transform of ϕ and of x_3 must meet transversally at 0. Now Lemma (3.1) shows that $k_j=1$.

If there were a different irreducible semi-invariant ρ whose highest-degree term is a multiple of, say, ϕ_{l_1}, then the corresponding stationary point at infinity would admit the (local) semi-invariants x_3, ϕ^*, and ρ^*. The zero sets of ϕ^* and ρ^* have (locally) only the point 0 in common, otherwise ϕ and ρ would have infinitely many common zeros, and hence be equal by Hilbert's Nullstellensatz. But then there must be at least three different irreducible local semi-invariants, a contradiction to (2.3). ∎

The basic argument in the proof works (and yields an upper bound for the degrees) whenever each stationary point at infinity admits only finitely many irreducible formal semi-invariants. See also Carnicer [5]. It should be remarked that the hypothesis of (3.4) involves only the highest degree term $f^{(m)}$, and homogeneous polynomial vector fields are easier to handle than general ones. We will discuss this later on. Theorem (3.4) also shows that the problem of existence (and construction, if applicable) of an algebraic integrating factor can be resolved whenever the hypothesis is satisfied. But more can be said about this:

THEOREM 3.5. *Assume that all the stationary points at infinity of* $\dot{x}=f(x)$ *are nondegenerate, that none of them is a rational node, and that the eigenvalue ratio for at least one of them is not rational.*

If $\dot{x}=f(x)$ *admits an integrating factor* $(\phi_1^{m_1/q}\cdots\phi_r^{m_r/q})^{-1}$, *then necessarily* $m_1=\cdots=m_r=q=1$, *and* $n_1+\cdots+n_r=m+1$. *Also,* $s=m+1$ *in the terminology of* (3.4), *and every* ψ_i *divides the highest-degree term of exactly one* ϕ_k, *with multiplicity one.*

Proof. Consider the Poincaré transform with respect to $\binom{1}{0}$. Since linear transformations do not change the number of irreducible polynomials, and the exponents, occurring in an integrating factor, we may assume that x_2 divides the highest-degree term of ϕ_1, and that the eigenvalue ratio at the corresponding stationary point at infinity is not a rational number. According to (3.3), $(x_3^{(m+2-(m_1n_1+\cdots+m_rn_r)/q)}\phi_1^{*(m_1/q)}\cdots\phi_r^{*(m_r/q)})^{-1}$ is then an integrating factor for f^*, and (2.4) shows that $\phi_2^*(0)\neq 0, ..., \phi_r^*(0)\neq 0$, and that $m_1/q=1$, and $m+2-(m_1n_1+\cdots+m_rn_r)/q=1$.

A similar argument applies to the factors of the highest-degree terms of the ϕ_i, for $i>1$: The Poincaré transforms of the remaining ϕ_j at the corresponding point at infinity are locally invertible, and the possible exponents are restricted by (2.4), resp. (2.3). But since x_3 is already known to have exponent -1, a case-by-case analysis of (2.4) shows that the Poincaré transform of ϕ_i must also have exponent -1. This shows $m_i/q=1$ for $i=1, ..., r$, and therefore $m_1=\cdots=m_r=q=1$. Considering the x_3-exponent again now shows $\sum n_i=m+1$.

Finally, no stationary point at infinity can have an integrating factor $x_3^{-1}\exp(\mu)$, as follows again from (2.4), and therefore every ψ_i must occur as highest-degree term of some ϕ_k. ▌

Thus, whenever f has no degenerate stationary point and no rational node at infinity then irreducible semi-invariants of f have degree $m+1$ or less, and every prime divisor of $\det(x, f^{(m)}(x))$ occurs at most once in a highest-degree term. It follows that finding irreducible semi-invariants for such a vector field is a problem of linear algebra: The coefficients of a semi-invariant satisfy a system of (finitely many) linear equations with parameters. (The fact that there are only finitely many possibilities for the highest-degree term helps to reduce work.) If the additional condition in (3.5) is satisfied (and this is always the case when none of the eigenvalue ratios equals zero; see (3.2)) then the search for an algebraic integrating factor (or deciding that no such factor exists) is even easier: There is, up to a scalar factor, at most one nonzero solution of $L_f(\phi)=\operatorname{div}(f)\,\phi$, and its highest-degree term is $\phi_{m+1}(x)=\det(x, f^{(m)}(x))$. Now finding $\phi_m, \phi_{m-1}, ...$ is a matter of solving a system of linear equations (without parameters). If this system has no solution then there is no algebraic integrating factor. In this context one should also mention the result of Kooij and Christopher [12], which complements (3.5) by describing the vector fields explicitly whenever a few more genericity conditions are satisfied.

Regarding the center problem (Schlomiuk [24, 25], Christopher [7]), we see that real equations with an integrating factor as above have the property that every stationary point with nonzero, purely imaginary eigenvalues of the linearization is a center; cf. (2.5).

It remains to discuss the conditions in (3.4) to see how restrictive they are. Since they involve only the highest-degree term, we have to discuss homogeneous polynomial vector fields of degree m. These form a finite dimensional vector (or affine) space $\mathscr{P}_m$. The vector fields p with the property that the entries p_1, p_2 are relatively prime form a Zariski-open and dense subset (by a resultant argument), and so do those vector fields for which $\det(x, p(x))$ has only prime factors of multiplicity 1. *We will always assume in the following that the vector fields under consideration are contained in the intersection $\mathscr{Q}_m$ of these two sets.* Thus for a given p, there are $m+1$ pairwise linearly independent vectors $v_1, ..., v_{m+1}$ such that $p(v_i)=\alpha_i v_i$, with $\alpha_i \neq 0$. Since $p(\sigma v_i)=\sigma^{m-1}\alpha_i(\sigma v_i)$, we may assume after scaling that $p(v_i)=v_i$. (Any nonzero v such that $p(v)=v$ will be called an *idempotent* of p.)

LEMMA 3.6. *Let $q \in \mathscr{Q}_m$, and λ an $(m-1)$-form such that p, defined by $p(x)=q(x)+\lambda(x)\cdot x$, also lies in $\mathscr{Q}_m$. Let $v \neq 0$ such that $q(v)=v$, and let m and β be the eigenvalues of $Dq(v)$.*

Then $1+\lambda(v)\neq 0$, and with $\sigma := (1+\lambda(v))^{-1/(m-1)}$ one has $p(\sigma v)=\sigma v$, and the eigenvalues of $Dp(\sigma v)$ are m and $\frac{\beta+\lambda(v)}{1+\lambda(v)}$.

The eigenvalue ratios of the linearizations at the stationary point at infinity corresponding to $\mathbf{C}v$ are $1-\beta$ for q, and $\frac{1-\beta}{1+\lambda(v)}$ for p.

Proof. The second assertion is an immediate consequence of the first, together with (3.1). As to the first, one has $p(v)=(1+\lambda(v))\,v$, hence $1+\lambda(v)\neq 0$ due to $p \in \mathscr{Q}_m$, and $p(\sigma v)=\sigma v$ follows. Furthermore, $Dp(x)\,y = Dq(x)\,y+\lambda(x)\,y+(D\lambda(x)\,y)\,x$, whence $Dq(v)\,w=\beta w+\gamma v$ (with v and w linearly independent) forces $Dp(v)\,w=(\beta+\lambda(v))\,w+\tilde{\gamma}v$. The assertion follows with $Dp(\sigma v)=(1+\lambda(v))^{-1}\,Dp(v)$. ∎

We will use this lemma as follows. Given $p \in \mathscr{Q}_m$, and $\rho(x):=\det(x, p(x))$, one has a decomposition

$$p(x)=\sum_{j=0}^{m+1}\alpha_j\begin{pmatrix} jx_1^{m+1-j}x_2^{j-1} \\ (j-1-m)\,x_1^{m-j}x_2^{j}\end{pmatrix}+\left(\sum_{i=0}^{m-1}\beta_i x_1^{m-i}x_2^{i}\right)\cdot x.$$

(This is easily verified.) Denote the first term by q, and the $(m-1)$-form in the second term by λ. Then $\operatorname{div}(q)=0$, and $\det(x, q(x))=\rho(x)$, and q is uniquely determined by these two properties. (We may assume that q is also contained in $\mathscr{Q}_m$, by choosing p out of a smaller, but still open-dense, subset of $\mathscr{P}_m$.) Note that λ has no influence on ρ.

Now let $v_1, ..., v_{m+1}$ be pairwise linearly independent idempotents of q. Due to $\operatorname{tr}(Dq(x))=0$ for all x, the eigenvalues of each $Dq(v_i)$ are m and $-m$. The eigenvalue ratio of the linearization at the corresponding stationary point at infinity is therefore $1+m$. For p, Lemma (3.6) yields the eigenvalue ratio $\frac{1+m}{1+\lambda(v_i)}$.

PROPOSITION 3.7. *There is a subset $\mathscr{Z}$ of $\mathscr{P}_m$ of Lebesgue measure zero so that every $p \in \mathscr{P}_m \backslash \mathscr{Z}$, and every polynomial vector field with highest-degree term p, satisfies the hypotheses of* (3.4) *and* (3.5).

Proof. Assume that, in the decomposition above, both p and q are in $\mathscr{Q}_m$. This only excludes a set of measure zero. Now let $\mathscr{Z}$ be the union of this measure zero set and the set containing those p for which $\alpha_0, ..., \alpha_{m+1}$, $\beta_0, ..., \beta_{m-1}$ are algebraically dependent over the rationals. (To see that the latter set has measure zero, recall that the set of nonzero polynomials in $2m+2$ variables with integer coefficients is countable, and that the set of zeros of any such polynomial has measure zero.)

The eigenvalue ratio for the linearization at the stationary point at infinity corresponding to $\mathbf{C}v_i$ is then $\frac{1+m}{1+\lambda(v_i)}$. If $p \notin \mathscr{Z}$ then the $\lambda(v_i)$ are not rational, since the components of v_i are algebraic over $\mathbf{C}(\alpha_0, ..., \alpha_{m+1})$, while none of the β_k is. ∎

This result is far from satisfactory, since the exceptional set contains, among others, all vector fields with rational coefficients. We therefore add a few more observations.

(3.8) PROPOSITION. *If p has rational coefficients, ρ is irreducible over the rational number field, and* $\operatorname{tr}(Dp(v)) \neq 0$ *for every idempotent of p, then p, and every polynomial vector field with highest-degree term p, satisfies the hypotheses of* (3.4) *and* (3.5).

Proof. Let $\hat{\rho}(s) := \rho(1, s)$. Then $\hat{\rho}$ is an irreducible polynomial in one variable, of degree $m+1$. If $\hat{\rho}(\vartheta) = 0$ then $p\binom{1}{\vartheta} = \alpha\binom{1}{\vartheta}$, with $\alpha = p_1(\vartheta)$ a polynomial of degree m in ϑ. (Note that every idempotent is a multiple of $\binom{1}{\vartheta}$, for a suitable ϑ, since x_2 does not divide ρ.)

Let β be the second eigenvalue of $Dp\binom{1}{\vartheta}$. It will be shown that the eigenvalue ratio $\frac{\alpha-\beta}{\alpha}$ at the corresponding stationary point at infinity is not rational. Otherwise, β would be a rational multiple of α, and the same would hold for the trace of $Dp\binom{1}{\vartheta}$, which is nonzero according to hypothesis. This trace is a polynomial in ϑ of degree smaller than m, and thus rationality of β/α implies that a certain (nonzero) polynomial of degree m annihilates ϑ. This is a contradiction, since $\hat{\rho}$ is the minimum polynomial of ϑ. ∎

As is well-known, the set of polynomial vector fields of given degree $m \geqslant 2$ that admit no algebraic integrating factor (actually, no algebraic semi-invariant) contains an open and dense subset of the space of all vector fields of degree m. See Zoladek [31] for a proof. The results above can be employed to construct such vector fields explicitly.

It may be useful to consider some examples of vector fields that admit an integrating factor as in (3.5). Let ϕ be a polynomial of degree $m+1$, such that its highest-degree term is a product of pairwise relatively prime linear forms, and $\phi = \phi_1 \cdots \phi_s$ its prime factorization. Let $q_i := \binom{-\partial\phi_i/\partial x_2}{\partial\phi_i/\partial x_1}$. (Note that the entries of q_i have no common factor, as the highest degree term of ϕ_i has no multiple prime factors.) Then for every choice of complex numbers $v_1, ..., v_s$, the vector field

$$f = \sum_{i=1}^{s} v_i \phi_1 \cdots \hat{\phi}_i \cdots \phi_s q_i$$

(with the hat indicating a missing term) admits the integrating factor ϕ^{-1}. As follows from Kooij and Christopher [12], under certain genericity conditions every vector field of degree $\leqslant m$ admitting the integrating factor ϕ^{-1} is of this type. It can also be shown that, unless all $v_i = 0$, the degree of f is exactly m.

To an integrating factor ϕ of this type, there exists a rational vector field g such that $\phi(x) = \det(g(x), f(x))$, and g then generates a local one-parameter group of symmetries of f; see [30]. Let us briefly disgress to see under what circumstances there exists a polynomial vector field with this property.

Proposition 3.9. *Let $f = \sum_{k=0}^{m} f^{(k)}$ be a vector field of degree $m > 1$ that satisfies the hypotheses of* (3.5), *and, in addition, assume that the entries of $f^{(m)}$ are relatively prime. If there is a polynomial vector field $g = \sum_{j=0}^{s} g^{(j)}$ with* $\det(g, f) \neq 0$ *and $[g, f] = \lambda f$ for some rational λ, then there is a $c \in \mathbf{C}^2$ so that $f(x) = f^{(m)}(x + c)$ for all x, and one may choose $g(x) = x + c$.*

Proof. The hypothesis implies that the entries of f are relatively prime. Therefore $[g, f] = \lambda f$, with g a polynomial, forces λ to be a polynomial. Moreover, the inverse of $\psi(x) := \det(g(x), f(x))$ is an integrating factor for f. It follows from (3.5) that ψ has degree $m+1$. Therefore the highest degree term $\det(g^{(s)}, f^{(m)})$ must be zero whenever $s > 1$. This condition, together with the relative primeness of the entries of $f^{(m)}$, forces $g^{(s)} = \sigma f^{(m)}$ with a suitable polynomial σ. But then $g - \sigma f$ has degree smaller than s, and $[g - \sigma f, f] = \tilde{\lambda} f$, with some polynomial $\tilde{\lambda}$.

Therefore we may assume that g has degree $\leqslant 1$. Since g cannot be constant (otherwise the entries of $f^{(m)}$ would be polynomials in one variable, contadicting the hypothesis), we have $g^{(1)} \neq 0$, and $\det(g^{(1)}(x), f^{(m)}(x))$ is the highest-degree term of ψ. But this highest-degree term is, up to a scalar factor, also equal to $\det(x, f^{(m)}(x))$, as it is an integrating factor of $f^{(m)}$. Therefore $\det(g^{(1)}(x) - \alpha x, f^{(m)}(x)) = 0$, and $g^{(1)}(x) - \alpha x = 0$, since it must be a polynomial multiple of $f^{(m)}$, and $m > 1$. Thus we may take $g(x) = x + c$.

Applying the polynomial automorphism $T: x \mapsto x - c$ to the equation $[x+c, f(x)] = \lambda(x) f(x)$ yields $[x, f(x-c)] = \lambda(x-c) f(x-c)$, and this forces $f(x-c)$ to be homogeneous. Since the highest degree term of $f(x-c)$ is $f^{(m)}(x)$, it follows that these two are equal, and thus $f(x) = f^{(m)}(x+c)$. ∎

In the terminology introduced before (3.9), this means that there is such a polynomial g if and only if ϕ is a product of factors of degree 1, and that their zero sets meet in one point. This indicates that nontrivial polynomial "infinitesimal symmetries" are rare even for those vector fields that admit an integrating factor with polynomial inverse, and that limiting the search to such vector fields would exclude many interesting ones.

As a class of examples, let us investigate quadratic vector fields more closely. There is a precise description of the exceptional set $\mathscr{X}$ in this case.

PROPOSITION 3.10. *Let p be a homogeneous quadratic vector field. If $p \in \mathscr{Q}_2$, and v_1, v_2 and v_3 are pairwise linearly independent idempotents of p, then there is a point $(\alpha_1 : \alpha_2 : \alpha_3)$ in the projective plane such that all $\alpha_i \neq 0$, $\alpha_1 + \alpha_2 + \alpha_3 \neq 0$, and $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$. This point $(\alpha_1 : \alpha_2 : \alpha_3)$ determines p up to a linear isomorphism, and conversely every point in the projective plane which satisfies the restrictions above determines a homogeneous quadratic vector field in $\mathscr{Q}_2$.*

For any quadratic vector field with highest-degree term p, the eigenvalue ratio of the linearization at the stationary point at infinity corresponding to $\mathbf{C}v_i$ is equal to $(\alpha_1 + \alpha_2 + \alpha_3)/\alpha_i$. In particular, if the normalization $\alpha_1 + \alpha_2 + \alpha_3 = 1$ is chosen then the vector field satisfies the hypotheses of (3.4) and (3.5) if and only if none of the α_i is rational and positive.

Proof. Some of the arguments are taken from [29]. For given p, defining $\hat{p}(x, y) := \frac{1}{2}(p(x+y) - p(x) - p(y))$ yields a symmetric, bilinear map from $\mathbf{C}^2 \times \mathbf{C}^2$ to $\mathbf{C}^2$, with $\hat{p}(x, x) = p(x)$ for all x. If c_1, c_2 is any basis of $\mathbf{C}^2$ then p is uniquely determined by $p(c_1)$, $p(c_2)$ and $\hat{p}(c_1, c_2)$, and conversely every set of prescribed values for these defines a homogeneous quadratic vector field. Now we have $p(v_1) = v_1$, $p(v_2) = v_2$, and the relation

$$
\begin{aligned}
-\frac{\alpha_1}{\alpha_3} v_1 - \frac{\alpha_2}{\alpha_3} v_2 = v_3 \\
&= \hat{p}(v_3, v_3) \\
&= \left(\frac{\alpha_1}{\alpha_3}\right)^2 \hat{p}(v_1, v_1) + 2\frac{\alpha_1 \alpha_2}{\alpha_3^2} \hat{p}(v_1, v_2) + \left(\frac{\alpha_2}{\alpha_3}\right)^2 \hat{p}(v_2, v_2)
\end{aligned}
$$

leads to

$$
2\hat{p}(v_1, v_2) = -\frac{\alpha_1 + \alpha_3}{\alpha_2} v_1 - \frac{\alpha_2 + \alpha_3}{\alpha_1} v_2.
$$

(Since the idempotents are pairwise linearly independent, it is clear that all $\alpha_i \neq 0$. The condition $\alpha_1 + \alpha_2 + \alpha_3 = 0$ yields $p(x) = \eta(x) \cdot x$ for some linear η, a contradiction to $p \in \mathcal{Q}_2$.)

Since $Dp(v_1)\, y = 2\hat{p}(v_1, y)$ for all y, this relation shows that the second eigenvalue of $Dp(v_1)$ (in addition to 2) is equal to $-\frac{(\alpha_2+\alpha_3)}{\alpha_1}$, and the assertion about the stationary point at infinity follows from (3.1). The rest is clear. ∎

Note that the inverse eigenvalue ratios at the stationary points at infinity add up to 1, as stated in (3.2). If one chooses α_1 and α_2 with nonzero imaginary parts and real parts $> 1/2$, and $\alpha_3 = 1 - \alpha_1 - \alpha_2$ then $(\alpha_1 : \alpha_2 : \alpha_3)$ corresponds to a point of $\mathcal{P}_2 \backslash \mathcal{Z}$. This argument shows that $\mathcal{P}_2 \backslash \mathcal{Z}$ contains a nonempty open subset of $\mathcal{P}_2$. Now it easily follows with (3.4) that there is a nonempty open subset in the space of all quadratic vector fields whose members admit no invariant algebraic curves. (A similar result can be deduced, with a little more work, for all degrees, and thus one has an elementary constructive proof of the corresponding result by Lins Neto [14], Theorem B.) In the quadratic case the idempotents are unique, while nontrivial scalar multiples (the scalars being roots of unity) of idempotents are again idempotents whenever $m > 2$. Let us look at one quadratic vector field where the hypothesis of (3.5) is not satisfied, to see that the method also may yield quite satisfactory results in such a case.

EXAMPLE 3.11. Let p be the homogeneous quadratic vector field determined by the idempotents v_1, v_2, v_3, and $\alpha_1 := -(1+i)$, $\alpha_2 := -(1-i)$, $\alpha_3 := 3$. Let f be a quadratic vector field with highest-degree term p, and assume that f admits an integrating factor $(\phi_1^{m_1/q} \cdots \phi_r^{m_r/q})^{-1}$, with pairwise relatively prime polynomials ϕ_i of respective degree n_i, and $m_1, ..., m_r, q$ relatively prime. Denote the zero set of the homogenization $\tilde{\phi}_i$ in the projective plane by Y_i, and the zero set of x_3 by Z. Furthermore, denote by w_i the stationary point at infinity corresponding to $\mathbf{C}v_i$. Then $r \leqslant 3$.

In case $r = 1$, one has $n_1 = 3$ and $m_1 = q = 1$.

In case $r = 2$, one has (up to labeling) $n_1 = 2$, $m_1/q = 1$, $n_2 = q$ and $m_2 = 1$. Moreover, if Y_1 and Z intersect at w_3 then $n_2 = q = 1$.

In case $r = 3$, one has (up to labeling) $n_1 = n_2 = 1$, and $m_1 = m_2 = n_3 = q$, and $m_3 = 1$. (Here, Y_3 and Z intersect in w_3.)

Proof. (i) It follows from (3.3) that the integrating factor at a stationary point at infinity equals $(x_3^s (\phi_1^*)^{m_1/q} \cdots (\phi_r^*)^{m_r/q})^{-1}$, where ".*" denotes the appropriate Poincaré transform, and $s = 4 - \sum m_i n_i / q$.

With the eigenvalue ratios from (3.10), and the results from (2.4) and (2.3), one gets a unique integrating factor $(\tilde{x}_2 x_3 \cdot (\text{inv.}))^{-1}$ (with a suitable definition of $\tilde{x}_2$, and "(inv.)" standing for an invertible factor) for the Poincaré transform corresponding to w_1, and in particular this implies

$s=1$, thus $\sum m_i n_i/q=3(*)$. Moreover, exactly one of the Y_i intersects Z at w_1, with multiplicity one. The same, mutatis mutandis, holds for w_2. Concerning the Poincaré transform corresponding to w_3, the integrating factor is $(x_3 \prod_{i\in I}(\tilde{x}_2+\beta_i x_3^3)^{d_i}\cdot(\text{inv.}))^{-1}$ (with suitable $\tilde{x}_2$, and using that $s=1$). (The integrating factor condition here requires $\sum d_i=1$.)

(ii) Suppose that Y_j and Z intersect at w_3, with multiplicity $e_j\geqslant 0$. Then locally $\phi_j^*=\prod_{i\in I_j}(\tilde{x}_2+\beta_i x_3^3)\cdot(\text{inv.})$, where I_j is a subset of I, of cardinality e_j. (Note that the intersection multiplicity of $(\tilde{x}_2+\beta_i x_3^3)$ and x_3 at $(0,0)$ is equal to 1!) Since $(\tilde{x}_2+\beta_i x_3^3)$ and $(\tilde{x}_2+\beta_l x_3^3)$, for $i\neq l$, intersect with multiplicity 3 at $(0,0)$, it follows that Y_j and Y_k, for $j\neq k$, intersect with multiplicity $3\cdot e_j\cdot e_k$, and it follows from Bezout's theorem that $3\cdot e_j\cdot e_k\leqslant n_j\cdot n_k$. Bezout's theorem also implies that Y_j and Z intersect at a total of n_j points (multiplicities counted), and the only possible intersection points are the w_i (since they must be stationary for the corresponding Poincaré transform of f). In particular, if Y_k intersects Z only at w_3 then $3\cdot e_j\cdot n_k\leqslant n_j\cdot n_k$, or $3\cdot e_j\leqslant n_j$. It follows that there cannot be two different Y_j that intersect Z only at w_3.

(iii) We may assume that Y_1 intersects Z at w_1, hence $m_1/q=1$. If Y_1 and Z also intersect at w_2, we have $n_1\geqslant 2$. If $n_1=2$ then $(*)$ implies $r\geqslant 2$, while $r\leqslant 2$ follows from (ii). Thus $r=2$, and $n_2 m_2/q=1$ with $(*)$. If $n_1>2$ then Y_1 and Z intersect at w_3 with multiplicity n_1-2. Again, (ii) shows that $r\leqslant 2$, and also that $3(n_1-2)\,n_2\leqslant n_1 n_2$ in case $r=2$, which forces $n_1=3$. But then $(*)$ forces $m_2 n_2/q=0$, a contradiction. Thus $r=1$, and $n_1=3$ by virtue of $(*)$.

If Y_1 and Z do not intersect at w_2 then we may assume that Y_2 and Z do. (Hence $m_2/q=1$.) Again, (ii) shows that $r\leqslant 3$, and Y_j and Z intersect at w_3 with multiplicity n_j-1, respectively, for $j=1,2$. From (ii) one finds $3(n_1-1)(n_2-1)\leqslant n_1 n_2$, or $3(1-\frac{1}{n_1})(1-\frac{1}{n_2})\leqslant 1$, which is impossible for $n_1\geqslant 2$ and $n_2\geqslant 2$. Thus we may assume that $n_2=1$. The case $n_1=1$ yields $r=3$, and $m_3 n_3/q=1$, while the case $n_1=2$, together with $(*)$, forces $r=2$. The assumption $n_1>2$ leads to $r=2$ (with (ii)), and then $n_1+n_2>3$ yields a contradiction to $(*)$. This finishes the proof. ∎

Concerning quadratic equations with algebraic particular integrals, one also has to mention the extensive investigations by Evdokimenko [8, 9, 10] on those equations which admit a cubic semi-invariant. In particular, Evdokimenko analyzed the global qualitative behavior of these equations in detail. The set of quadratic equations admitting a cubic integrating factor is contained in this class.

As an other illustration that the tools developed here work also for certain dicritical stationary points, we discuss "generic" second-order equations in one variable.

EXAMPLE 3.12. Let γ be a polynomial in two variables, of degree $m>1$, such that the highest degree term $\gamma^{(m)}$ satisfies $\gamma^{(m)}(1,0)\neq 0$, $\gamma^{(m)}(0,1)\neq 0$, and $\gamma^{(m)}$ has only simple prime factors. Then the equation

$$\ddot{x}=\gamma(\dot{x},x),$$

resp. the equivalent two-dimensional system, has no algebraic integrating factor whenever $m>2$. If $m=2$ and there is an algebraic integrating factor then there also is an integrating factor of the form ψ^{-1}, with ψ a quadratic polynomial satisfying $\psi^{(2)}=\gamma^{(2)}$.

Proof. The equivalent system may be written as $\dot{x}=f(x)$, with

$$f(x)=\begin{pmatrix}\gamma(x)\\ x_1\end{pmatrix}.$$

Therefore $\det(x,f^{(m)}(x))=x_2\cdot\gamma(x)$, and the stationary points at infinity correspond to $\binom{1}{0}$ and to the homogeneous zeros of γ. Now assume that f admits an algebraic integrating factor of the form (†), with ϕ_i of degree n_i.

(i) Consider the Poincaré transform with respect to $\binom{1}{0}$, and let $\gamma(1,0)=\alpha$. Then the hypothesis implies $\alpha\neq 0$, and a routine computation similar to the one in the proof of (3.1) shows

$$f^*(x_2,x_3)=\begin{pmatrix}-\alpha x_2+x_2x_3(\cdots)+x_3^{m-1}\\ -\alpha x_3+x_3^2(\cdots)\end{pmatrix}.$$

Thus for $m>2$ this stationary point at infinity is dicritical with eigenvalue ratio 1, while for $m=2$ there is only one irreducible semi-invariant, viz. x_3.

Now assume $m>2$. If $\phi_i^*(0)=0$ then ϕ_i^* is a product of v_i irreducible formal semi-invariants, where $1\leqslant v_i\leqslant n_i$. (Note that no local component of the zero set of $\tilde{\phi}_i$ is tangent to $x_3=0$, as can be seen from (2.3). Then $v_i\leqslant n_i$ follows from the definition of the intersection multiplicity and Bezout's theorem.) Let $s\geqslant 0$ such that $\phi_i^*(0)=0$ if and only if $i\leqslant s$ (after a suitable permutation, if necessary), and define $v_i:=0$ for $i>s$. Since

$$\left(x_3^{(m+2-(m_1n_1+\cdots+m_rn_r)/q)}\phi_1^{*(m_1/q)}\cdots\phi_r^{*(m_r/q)}\right)^{-1}$$

is an integrating factor of f^*, (2.3b) yields

$$m+2-\sum_{i=1}^{r}m_in_i/q+\sum_{i=1}^{r}m_iv_i/q=2,\text{ resp.}$$

$$m=\sum_{i=1}^{r}m_i(n_i-v_i)/q. \qquad (*)$$

(ii) For the remaining stationary points at infinity we use the computations carried out in [23], Lemma 4.2. As was shown there, the first relevant terms of the normal form (in suitable coordinates) are given by

$$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}^{\cdot} = \begin{pmatrix} \beta x_2 + \cdots \\ \delta x_3^m + \cdots \end{pmatrix}, \qquad \text{with} \quad B_s(x) = \begin{pmatrix} \beta x_2 \\ 0 \end{pmatrix},$$

and nonzero β, δ. Therefore one has a unique local integrating factor $(x_2 x_3^m)^{-1}$. Thus, if $\mathbf{C}v \neq \mathbf{C}\binom{1}{0}$ is a stationary point at infinity, and $\phi^*_{i,v}(0) = 0$, then $\phi^*_{j,v}(0) \neq 0$ for all other indices j, and $m_i/q = 1$. Moreover, the x_3-exponent yields $m = m + 2 - \sum m_i n_i/q$, thus

$$\sum_{i=1}^{r} m_i n_i/q = 2. \tag{$**$}$$

Combining this with (i), we see that $v_i < n_i$ implies $m_i/q = 1$, and this leads to an improved version of $(*)$:

$$\sum_{i=1}^{r} (n_i - v_i) = m. \tag{$***$}$$

(iii) If every ϕ_i satisfies $\phi^*_{i,v}(0) = 0$ for some $v \notin \mathbf{C}\binom{1}{0}$ then $m_i/q = 1$ for all i, and comparing $(**)$ and $(***)$ yields $m = \sum (n_i - v_i) \leqslant \sum n_i = 2$, hence $m = 2$ and all $v_i = 0$. Therefore, if $m > 2$ there is a semi-invariant ψ such that the homogenized polynomial $\tilde{\psi}$ intersects $x_3 = 0$ only at $(1:0:0)$. With $s := \deg(\psi)$ we may assume that $\psi^{(s)} = x_2^s$.

(iv) We will now show that such a semi-invariant cannot exist. Compare terms of highest degree in $L_f(\psi) = \lambda\psi$ (with λ some polynomial) to find $L_{f^{(m)}}(\psi^{(s)}) = \lambda^{(m-1)}\psi^{(s)}$. Now $f^{(i)} = \binom{*}{0}$ for all $i > 1$ shows that $L_{f^{(i)}}$ sends every polynomial that does not depend on x_1 to zero. Therefore $L_{f^{(m)}}(\psi^{(s)}) = 0$, and $\lambda^{(m-1)} = 0$. A simple induction, comparing terms of descending degree, and using that x_2 does not divide $\gamma^{(s)}$, shows that for all d such that $m - d > 1$ one has $\lambda^{(m-d)} = 0$ and $\partial\psi^{(s-d+1)}/\partial x_1 = 0$. Comparing the terms for $m - d = 1$ one obtains

$$s x_1 x_2^{s-1} + 0 + \gamma^{(m)} \frac{\partial \psi^{(s-d)}}{\partial x_1} = \lambda^{(m-d-1)} x_2^s.$$

From this follows that either x_2 divides $\gamma^{(m)}$ or that $s x_1 x_2^{s-1} = \lambda^{(m-d-1)} x_2^s$, and there is a contradiction in both cases.

(v) If $m = 2$ then $v_i = 0$ and $m_i/q = 1$ for all i is known from (iii). Then $(***)$ shows $r \leqslant 2$, and one obtains an integrating factor ψ^{-1}, with ψ

either an irreducible quadratic polynomial or a product of two polynomials of degree one. The last assertion follows from (ii) and (iii), since the homogenized polynomial $\tilde{\psi}$ must intersect the line $x_3 = 0$ in the same points as $\tilde{\gamma}$. ▮

There are nontrivial quadratic equations with an integrating factor as stated in (3.12): If τ is a polynomial in one variable of degree at most two, and α, β are nonzero constants, then the equation

$$\dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} \alpha\tau'(x_2) + \beta(-(1/2\alpha)\, x_1^2 + \tau(x_2)) \\ x_1 \end{pmatrix}$$

admits the integrating factor ϕ^{-1}, with $\phi(x) = (-1/2\alpha)\, x_1^2 + \tau(x_2)$. Using [30], Prop. 3.2, one can show that ϕ and the vector field necessarily have this form whenever ϕ is irreducible. If ϕ is reducible then it is also possible to determine the equations with integrating factor ϕ^{-1}, using results of Kooij and Christopher [12].

As these examples show, the existence of dicritical stationary points at infinity does not automatically imply that the corresponding vector fields and possible integrating factors are beyond reach. It requires additional work (and additional tools) to discuss them, but this can be done successfully in several instances. One may hope that future developments will yield a complete picture.

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